Tutorial 7 MAU22E01

1 Gram - Schmidt Process

Convert a basis $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ into an orthogonal basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

Step 1 : $\bar{v}_1 = \bar{u}_1$

Step 2 : $\bar{v}_2 = \bar{u}_2 - \dfrac{\langle \bar{u}_2, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1$

Step 3 : $\bar{v}_3 = \bar{u}_3 - \dfrac{\langle \bar{u}_3, \bar{v}_1 \rangle}{\|v_1\|^2} \bar{v}_1 - \dfrac{\langle \bar{u}_3, \bar{v}_2 \rangle}{\|v_2\|^2} \bar{v}_2$

1(i) $\bar{u}_1 = (0,1)$ $\bar{u}_2 = (2,-3)$

Taking the dot product as our inner product

Step 1 : $\bar{v}_1 = \bar{u}_1 = (0,1)$

Step 2 : $\bar{v}_2 = \bar{u}_2 - \dfrac{\langle \bar{u}_2, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1$

$\Rightarrow \bar{v}_2 = (2,-3) - \dfrac{\langle (2,-3), (0,1) \rangle}{\langle (0,1), (0,1) \rangle} (0,1)$

$\bar{v}_2 = (2,-3) - \dfrac{2(0) + (-3)(1)}{(0)(0) + (1)(1)} (0,1)$

$$\bar{V}_2 = (2,-3) - \left(\frac{-3}{1}(0,1) \right)$$

$$\bar{V}_2 = (2,-3) + (0,3)$$

$$\bar{V}_2 = (2,0)$$

- The basis $\{(0,1),(2,0)\}$ is orthogonal with respect to the dot product.

1(ii) $\bar{u}_1 = (1,0,1)$, $\bar{u}_2 = (0,0,1)$, $\bar{u}_3 = (2,-1,0)$

Taking the dot product as our inner product.

- Step 1 : $\bar{V}_1 = \bar{u}_1 = (1,0,1)$

Step 2 : $\bar{V}_2 = \bar{u}_2 - \frac{\langle \bar{u}_2, \bar{V}_1 \rangle}{\|\bar{V}_1\|^2} \bar{V}_1$

$$\Rightarrow \bar{V}_2 = (0,0,1) - \frac{\langle (0,0,1),(1,0,1) \rangle}{\langle (1,0,1),(1,0,1) \rangle}(1,0,1)$$

$$\bar{V}_2 = (0,0,1) - \frac{(0)(1)+(0)(0)+(1)(1)}{(1)(1)+(0)(0)+(1)(1)}(1,0,1)$$

$$\bar{V}_2 = (0,0,1) - \frac{1}{2}(1,0,1)$$

$$\bar{V}_2 = (0,0,1) + \left(-\frac{1}{2}, 0, -\frac{1}{2}\right)$$

$$\bar{V}_2 = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)$$

- Step 3 : $\bar{V}_3 = \bar{u}_3 - \frac{\langle \bar{u}_3, \bar{V}_1 \rangle}{\|V_1\|^2}\bar{V}_1 - \frac{\langle \bar{u}_3, \bar{V}_2 \rangle}{\|V_2\|^2}\bar{V}_2$

$$\Rightarrow \bar{V}_3 = (2,-1,0) - \frac{\langle (2,-1,0),(1,0,1) \rangle}{\langle (1,0,1),(1,0,1) \rangle}(1,0,1)$$

$$- \frac{\langle (2,-1,0),\left(-\frac{1}{2},0,\frac{1}{2}\right) \rangle}{\langle \left(-\frac{1}{2},0,\frac{1}{2}\right),\left(-\frac{1}{2},0,\frac{1}{2}\right) \rangle}\left(-\frac{1}{2},0,\frac{1}{2}\right)$$

$$\Rightarrow \bar{V}_3 = (2,-1,0) - \left(\frac{(2)(1)+(-1)(0)+(0)(1)}{(1)(1)+(0)(0)+(1)(1)}\right)(1,0,1)$$

$$- \left(\frac{(2)\left(-\frac{1}{2}\right)+(-1)(0)+(0)\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+(0)(0)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}\right)\left(-\frac{1}{2},0,\frac{1}{2}\right)$$

$$\Rightarrow \bar{V}_3 = (2,-1,0) - \left(\frac{2}{2}\right)(1,0,1) - \left(\frac{-1}{0.5}\right)\left(-\frac{1}{2},0,\frac{1}{2}\right)$$

$$\bar{V}_3 = (2,-1,0) + (-1,0,-1) + (-1,0,1)$$

$\vec{v}_3 = (0, -1, 0)$

The basis $\{(1,0,1), (\frac{-1}{2}, 0, \frac{1}{2}), (0,-1,0)\}$ is orthogonal with respect to the dot product.

Ex 2

Given matrix A, the characteristic polynomial of A, $P(\lambda) = \det(\lambda I - A)$

where I is the identity matrix and λ is any eigenvalue of A.

2(i) $A = \begin{pmatrix} -1 & 5 \\ 0 & 1 \end{pmatrix}$

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 5 \\ 0 & 1 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -1 & 5 \\ 0 & 1 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda + 1 & -5 \\ 0 & \lambda - 1 \end{pmatrix}$$

The characteristic polynomial of A

$$P(\lambda) = \det \begin{bmatrix} \lambda + 1 & -5 \\ 0 & \lambda - 1 \end{bmatrix}$$

$$P(\lambda) = (\lambda+1)(\lambda-1) - (-5)(0)$$

$$P(\lambda) = (\lambda+1)(\lambda-1) = \lambda^2 - 1$$

The eigenvalues of A are given by the equation:

$$P(\lambda) = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$\Rightarrow$ The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = +1$

(you aren't asked to find the eigenvalues)

2 (ii) $A = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 2 \\ 1 & \lambda \end{pmatrix}$$

The characteristic polynomial of A,

$$P(\lambda) = \det \begin{bmatrix} \lambda & 2 \\ 1 & \lambda \end{bmatrix}$$

$P(\lambda) = (\lambda)(\lambda) - (2)(1)$

$P(\lambda) = \lambda^2 - 2$

The eigenvalues of A are given by the equation:

$P(\lambda) = 0$

$\Rightarrow \lambda^2 - 2 = 0$

$\lambda^2 = 2$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 2$.

(you aren't asked to find the eigenvalues.)

2(iii) $A = \begin{pmatrix} -1 & 2 & 1 & 3 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$

$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 2 & 1 & 3 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$

$\lambda I - A = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} -1 & 2 & 1 & 3 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$

$$\lambda I - A = \begin{pmatrix} \lambda+1 & -2 & -1 & -3 \\ 0 & \lambda-2 & 2 & 1 \\ 0 & 0 & \lambda-3 & 0 \\ 0 & 0 & -2 & \lambda+1 \end{pmatrix}$$

The characteristic polynomial of A,

$$P(\lambda) = \det \begin{bmatrix} \lambda+1 & -2 & -1 & -3 \\ 0 & \lambda-2 & 2 & 1 \\ 0 & 0 & \lambda-3 & 0 \\ 0 & 0 & -2 & \lambda+1 \end{bmatrix}$$

(As given in the supplemental lecture)

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for co-factor expansion across the j^{th} column.

Where a_{ij} is the entry in the i^{th} row, j^{th} column of matrix A.

And M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix that results from A by removing the i^{th} row and j^{th} column of A.

Expanding along the $j=1$ column

$$P(\lambda) = (-1)^{1+1}(\lambda+1)\det\begin{bmatrix} \lambda-2 & 2 & 1 \\ 0 & \lambda-3 & 0 \\ 0 & -2 & \lambda+1 \end{bmatrix}$$

$$+ (-1)^{1+2}(0)\det\begin{bmatrix} -2 & -1 & -3 \\ 0 & \lambda-3 & 0 \\ 0 & -2 & \lambda+1 \end{bmatrix}$$

$$+ (-1)^{1+3}(0)\det\begin{bmatrix} -2 & -1 & -3 \\ \lambda-2 & 2 & 1 \\ 0 & -2 & \lambda+1 \end{bmatrix}$$

$$+ (-1)^{1+4}(0)\det\begin{bmatrix} -2 & -1 & -3 \\ \lambda-2 & 2 & 1 \\ 0 & \lambda-3 & 0 \end{bmatrix}$$

$$P(\lambda) = (\lambda+1)\det\begin{bmatrix} \lambda-2 & 2 & 1 \\ 0 & \lambda-3 & 0 \\ 0 & -2 & \lambda+1 \end{bmatrix} + 0 + 0 + 0$$

Doing co-factor expansion along the $j=1$ column again.

$$P(\lambda) = (\lambda+1)(-1)^{1+1}(\lambda-2)\det\begin{bmatrix} \lambda-3 & 0 \\ -2 & \lambda+1 \end{bmatrix}$$

$$+ (\lambda+1)(-1)^{1+2}(0)\det\begin{bmatrix} 2 & 1 \\ -2 & \lambda+1 \end{bmatrix}$$

$$+ (\lambda+1)(-1)^{1+3}(0)\det\begin{bmatrix} 2 & 1 \\ \lambda-3 & 0 \end{bmatrix}$$

$$P(\lambda) = (\lambda+1)(\lambda-2)\left[(\lambda-3)(\lambda+1) - (0)(-2)\right]$$

$$P(\lambda) = (\lambda+1)(\lambda-2)(\lambda-3)(\lambda+1) + 0$$

$$P(\lambda) = (\lambda^2 - \lambda - 2)(\lambda^2 - 2\lambda - 3)$$

$$P(\lambda) = \lambda^4 - 2\lambda^3 - 3\lambda^2 - \lambda^3 + 2\lambda^2 + 3\lambda$$
$$- 2\lambda^2 + 4\lambda + 6$$

$$P(\lambda) = \lambda^4 - 3\lambda^3 - 3\lambda^2 + 7\lambda + 6$$

2(iv) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda-1 & -2 & 0 \\ 0 & \lambda-1 & -2 \\ -1 & 1 & \lambda-1 \end{pmatrix}$$

The characteristic polynomial of A,

$$P(\lambda) = \det \begin{bmatrix} \lambda-1 & -2 & 0 \\ 0 & \lambda-1 & -2 \\ -1 & 1 & \lambda-1 \end{bmatrix}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for co-factor expansion along the j^{th} column

Expanding along the $j=1$ column

$$P(\lambda) = (-1)^{1+1}(\lambda-1)\det\begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-1 \end{bmatrix}$$

$$+ (-1)^{1+2}(0)\det\begin{bmatrix} -2 & 0 \\ 1 & \lambda-1 \end{bmatrix}$$

$$+ (-1)^{1+3}(-1)\det\begin{bmatrix} -2 & 0 \\ \lambda-1 & -2 \end{bmatrix}$$

$$P(\lambda) = (\lambda-1)\left[(\lambda-1)(\lambda-1) - (-2)(1)\right]$$

$$+ 0 + (-1)\left[(-2)(-2) - (0)(\lambda-1)\right]$$

$$P(\lambda) = (\lambda-1)(\lambda-1)(\lambda-1) + 2(\lambda-1) - 4 + 0$$

$$P(\lambda) = (\lambda^2 - 2\lambda + 1)(\lambda-1) + 2\lambda - 2 - 4$$

$$P(\lambda) = \lambda^3 - \lambda^2 - 2\lambda^2 + 2\lambda + \lambda - 1 + 2\lambda - 6$$

$$P(\lambda) = \lambda^3 - 3\lambda^2 + 5\lambda - 7$$